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Existence of Best Approximations

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A simple, yet general, existence theorem for best approximations is proved. It contains the majority of the known existence theorems.

1. INTRODUCTION

One of the basic questions in approximation theory concerns the existence of best approximations. Specifically, let K be a subset of a normed linear space X and let $x \in X$. The (possibly empty) set of best approximations to x from K is defined by

$$P_K(x) = \{y \in K \mid \| x - y \| = d(x, K)\},$$

where $d(x, K) = \inf\{\| x - y \| \mid y \in K\}$. The set K is called *proximinal* (resp. *Chebyshev*) if $P_K(x)$ contains at least (resp. exactly) one point for every $x \in X$. The mapping $P_K : X \to 2^K$ is called the *metric projection* onto K.

In this terminology, the basic existence question can be phrased as: Which subsets are proximinal?

There is much that is known concerning existence of best approximations. The following list of examples is intended to be a representative sampling of some of the more useful known proximinal sets.

(1) Any reflexive subspace (Klee [19]), e.g., a finite-dimensional subspace (Riesz [23]).

(2) Any weak* closed subset of a dual space (Phelps [21]).

(3) Any closed convex subset of a reflexive space (Klee [19]).

(4) The rational functions $\mathscr{R}_n^m$ in $L_p[a, b]$, $1 \leqslant p \leqslant \infty$ (Walsh [31], Efimov and Stechkin [10]).

(5) The exponential sums in $L_p[a, b]$, $1 \leqslant p \leqslant \infty$ (Hobby and Rice [15], de Boor [5], Werner [32], Kammler [18]).

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(6) The splines in $C[a, b]$ of order n with k free knots (Schumaker [25]).

(7) Any weak-operator closed subset of the space of operators on a Hilbert space, e.g., the positive or Hermitian operators (Halmos [13]).

The known proofs of these results exhibit a variety of techniques, although there is a common thread of "compactness" interwoven throughout.

In this paper we will prove a simple yet general existence theorem which includes as special cases *all* of the examples mentioned above. To do this, we will first generalize the important notion of an "approximatively compact" set (which was introduced by Efimov and Stechkin [10] and later extended to "approximatively weakly compact" by Breckner [3]) to what we call "approximatively τ-compact" for a "regular mode of convergence τ" (see definitions in Section 2). Each approximatively τ-compact set is easily seen to be proximinal and, moreover, its metric projection satisfies a certain continuity condition. In particular, each of the examples (1)–(7) mentioned above turns out to be an approximatively τ-compact set for an appropriate mode of convergence τ. In practice, τ is usually taken to be convergence relative to the norm, weak, or weak* topologies. However, in some of the most interesting examples (e.g., in $C[a, b]$, the rational functions, exponential sums, or splines with free knots), τ does *not* arise from any topology on the space (cf. Theorems 3.1 and 3.3 below).

In Section 2 we state the main definitions and obtain some results of a general nature. We also introduce there property (A_τ): if a net (x_δ) τ-converges to x and $\| x_\delta \| \to \| x \|$, then $\| x_\delta - x \| \to 0$. (This generalizes a well-known geometric property of Fan and Glicksberg [11] when τ is generated by the weak topology.) In spaces with property (A_τ), every approximatively τ-compact set is actually approximatively compact (Theorem 2.15). One consequence of this is that every weak* closed subset of a locally uniformly convex dual space is approximatively compact (Corollary 2.17). Also (Proposition 2.24) an approximation theoretic characterization of those dual spaces having property (A_τ) is given, where τ denotes weak* sequential convergence. In Section 3 we consider some applications in $C[a, b]$ and show that the (generalized) rational functions, splines with free knots, and exponential sums are approximatively Δ-compact. In Section 4, the rational functions and exponential sums are considered as subsets of $L_p[a, b]$ for $1 \leqslant p \leqslant \infty$. In Section 5 we observe (Theorem 5.2) that every weak*-operator closed subset of the space of bounded linear operators $\mathscr{L}(X, Z^*)$ from X into Z^* is approximatively weak* operator-compact. This implies that example (7) above is proximinal. In Section 6 we consider some further extensions possible by "localizing" the definition of approximatively τ-compact (Theorem 6.1). As a particular application, we recover an existence theorem of Dunham [9] (see the remark following Proposition 6.4).

Except where explicitly noted otherwise, all of the main ideas and essentially all of the results presented here were obtained during September–October, 1972, and were included in my mimeographed lecture notes [6] written during this period. Since then, a few other writers have considered some similar things. In particular, Vlasov [30] has also defined an approximatively τ-compact set, but he required, in addition, that τ be a *topology*. However, this requirement apparently excludes some of the more interesting applications which are included under our definition (e.g., the rational functions, exponential sums, and splines with free knots; see Theorems 3.1 and 3.3).

Some of the results in this paper were presented, without proofs, in [7]. In addition, the subspace case was considered in further detail in [7] (but not here).

All undefined terms or results are standard and can be found, for example, in [8].

2. General Results

We first define a general type of convergence for nets or sequences in a normed space.

2.1. Definition. Let X be a normed linear space. Suppose that in X certain nets (resp. sequences) are said to *τ-converge*, written $x_\delta \to^\tau x$. Suppose also that this convergence has the following properties.

(i) τ is "translation invariant," i.e., $x_\delta \to^\tau x$ implies $x_\delta + y \to^\tau x + y$ for any $y \in X$.

(ii) τ is "norm dominated," i.e., $x_\delta \to^\tau x$ implies $\|x\| \leq \limsup \|x_\delta\|$.

(iii) τ is "homogeneous," i.e., $x_\delta \to^\tau x$ implies $\alpha x_\delta \to^\tau \alpha x$ for every scalar α.

In this situation τ is called a *regular mode of convergence* (resp. *sequential convergence*) on X.

Clearly, every regular mode of convergence is a regular mode of sequential convergence, but the converse is not true in general.

2.2. Examples. Unless otherwise stated, each of the following is a regular mode of convergence. The notation we adopt in these examples will be used throughout the paper.

2.2.1. *Convergence in norm*: $x_\delta \to^{n} x$ iff $\|x_\delta - x\| \to 0$.

2.2.2. *Weak convergence*: $x_\delta \to^{w} x$ iff $x^*(x_\delta) \to x^*(x)$ for each $x^* \in X^*$.

2.2.3. If $X = Y^*$ is a dual space, *weak* convergence* in X: $y_\delta^* \to^{w^*} y^*$ iff $y_\delta^*(y) \to y^*(y)$ for each $y \in Y$.

2.2.4. In the space $C(T)$ of continuous functions on a compact Hausdorff space T, *pointwise convergence on a dense subset* of T: $x_\delta \to^{\Delta} x$ iff there is a dense subset $T_0 = T_0(x_\delta, x)$ of T such that $x_\delta(t) \to x(t)$ for each $t \in T_0$.

2.2.5. In $C[a, b]$, *pointwise convergence at all except possibly finitely many points*: $x_\delta \to^{\phi} x$ iff there is a set $T_0 = T_0(x_\delta, x)$ in $[a, b]$ such that $[a, b] \backslash T_0$ is finite and $x_\delta(t) \to x(t)$ for every $t \in T_0$.

2.2.6. In an $L_p(\mu)$ space $(1 \leqslant p \leqslant \infty)$, *convergence of a sequence almost everywhere*: $x_n \to^{\text{a.e.}} x$ iff $\mu(\{t \mid x_n(t) \nrightarrow x(t)\}) = 0$. (This is a regular mode of *sequential* convergence—using Fatou's lemma—but it is not, in general, a regular mode of convergence. To see this, consider the net (x_δ) of all characteristic functions of finite sets in [0, 1] ordered by containment. Then $x_\delta(t) \to 1$ for every $t \in [0, 1]$. However, $\| x_\delta \|_p = 0$ for every δ and $\| 1 \|_p = 1$. Thus 2.1(ii) fails.)

2.2.7. *Almost weak convergence*: $x_\delta \to^{\text{aw}} x$ iff there is a weak* dense subset Λ of the set of extreme points of the unit ball in X^* such that $x^*(x_\delta) \to x^*(x)$ for each $x^* \in \Lambda$. (Note that in the space $C(T)$, aw-convergence coincides with Δ-convergence as defined in 2.2.4.)

In the remaining examples, we consider regular modes of convergence in the space $\mathscr{L}(X, Y)$ of all bounded linear operators L from the normed linear space X into Y with the norm

$$\| L \| = \sup_{x \neq 0} \frac{\| L(x) \|}{\| x \|}.$$

The properties below are listed in order of increasing generality.

2.2.8. *Uniform convergence:* $L_\delta \to^{\text{u}} L$ iff $\| L_\delta - L \| \to 0$.

2.2.9. *Strong convergence:* $L_\delta \to^{\text{s}} L$ iff $\| L_\delta(x) - L(x) \| \to 0$ for each $x \in X$.

2.1.10. *Weak operator convergence:* $L_\delta \to^{\text{wo}} L$ iff $y^*(L_\delta(x)) \to y^*(L(x))$ for every $x \in X$ and $y^* \in Y^*$.

2.2.11. *Weak* operator convergence* if $Y = Z^*$ is a dual space: $L_\delta \to^{\text{w*o}} L$ iff $L_\delta(x)(z) \to L(x)(z)$ for each $x \in X$ and $z \in Z$.

Note that if Y is reflexive, then wo-convergence and w*o-convergence in $\mathscr{L}(X, Y)$ coincide.

If τ is a regular mode of convergence on X, we write τs for the induced regular mode of sequential convergence on X. For example, $x_n \to^{\text{ws}} x$ if $x_n \to^{\text{w}} x$.

A regular mode of convergence (resp. sequential convergence) τ is called *topological* if there is a topology on X such that convergence of a net (resp. sequence) in this topology is equivalent to τ-convergence. Examples 2.2.1–

2.2.3 and 2.2.8–2.2.11 are topological. However, examples 2.2.4–2.2.7 are not topological. (In each case the proof of this fact follows by constructing a sequence which does not τ-converge to zero yet every subsequence has a subsequence which does τ-converge to zero. Such a construction would be impossible if τ were topological.)

2.3. DEFINITION. Let K be a subset of the normed space X and $x \in X$. A sequence (y_n) in K is called a *minimizing sequence* for x if $\| x - y_n \| \to d(x, K)$. Let τ be a regular mode of convergence (resp. sequential convergence) on X. The set K is called *approximatively* τ-compact if for each $x \in X$, each minimizing sequence for x has a subnet (resp. subsequence) which τ-converges to a point of K.

For $\tau = n$ (i.e. convergence relative to the norm topology), such sets were first studied by Efimov and Stechkin [10] and called *approximatively compact*. (We will usually defer to their simpler terminology for $\tau = n$; i.e., approximatively compact and approximatively norm-compact are synonymous.) Breckner [3] generalized approximative compactness by replacing the norm with the weak topology. That is, he considered approximatively w-compact sets. It should be mentioned that the original motivation for introducing approximative compactness in [10] was to aid in the study of the problem of the convexity of Chebyshev sets, and not existence problems.

The main reason for the introduction of approximatively τ-compact sets is that they are proximinal and their metric projections satisfy a certain continuity criterion (see Theorem 2.7).

A property somewhat stronger than approximative τ-compactness is bounded τ-compactness.

2.4. DEFINITION. Let τ be a regular mode of convergence (resp. sequential convergence) on X. A subset K of X is called *boundedly τ-compact* if each bounded net (resp. sequence) has a subnet (resp. subsequence) which τ-converges to a point of K.

The particular cases of bounded τ-compactness when τ denotes convergence relative to either the norm or weak topologies were first mentioned by Klee [19] who showed such sets were proximinal. More generally, it follows easily from the definition that

2.5. LEMMA. *Every boundedly τ-compact set is approximatively τ-compact.*

In addition to approximatively τ-compact sets being proximinal, their metric projections also satisfy a certain continuity criterion. To show this, we first must define the notions of τ-open, τ-compact, etc.

2.6. DEFINITION. Let τ be a regular mode of convergence (resp. sequential convergence) on X.

(1) A subset F of X is called *τ-closed* if F contains the limit of each of its τ-convergent nets (resp. sequences).

(2) A subset G of X is called *τ-open* if $X\backslash G$ is τ-closed.

(3) A subset A of X is called *countably τ-compact* if each sequence in A has a subnet (resp. subsequence) which τ-converges to a point in A.

(4) A subset A of X is called *τ-compact* if each net (resp. sequence) in A has a subnet (resp. subsequence) which τ-converges to a point in A.

If τ is topological, then the above definitions are equivalent to the usual (i.e., topological) ones.

(5) The metric projection onto a set K is called *norm-τ upper semicontinuous* (briefly, norm-τ u.s.c.) *at a point* x_0 provided that for each sequence (x_n) with $\| x_n - x_0 \| \to 0$ and each τ-open set $V \supset P_K(x_0)$, we have $V \supset P_K(x_n)$ eventually (i.e., for n sufficiently large). P_K is called *norm-τ u.s.c.* if it is norm-τ u.s.c. at each point of X.

If τ is generated by the norm topology, then norm-τ u.s.c. reduces to the usual notion of upper semicontinuity (u.s.c.) for set-valued maps (cf. Hahn [12]). If τ is topological and P_K is singleton-valued (i.e., K is Chebyshev), then norm-τ u.s.c. reduces to ordinary continuity of the mapping P_K from X with its norm topology into K with its τ topology.

The fundamental properties of approximatively τ-compact sets can now be stated.

2.7. THEOREM. *Let τ be a regular mode of convergence (resp. sequential convergence) on X, and let K be an approximatively τ-compact set. Then*

(1) *K is proximinal;*

(2) *P_K is norm-τ upper semicontinuous;*

(3) *$P_K(x)$ is countably τ-compact for each $x \in X$.*

Moreover, if K is boundedly τ-compact then (1), (2), *and the following statement hold.*

(4) *$P_K(x)$ is τ-compact for each $x \in X$.*

Proof. (1) Let $x \in X$ and let (y_n) be a minimizing sequence in K for x. Then (y_n) has a subnet (resp. subsequence) (y_δ) which τ-converges to some $y_0 \in K$. Then by 2.1(i) and (ii), $y_\delta - x \to^\tau y_0 - x$ and

$$\| y_0 - x \| \leqslant \limsup \| y_\delta - x \| = d(x, K).$$

Thus $y_0 \in P_K(x)$ and K is proximinal.

(2) If the result is false, there exist $x_0 \in X$, a sequence (x_n) with $\| x_n - x_0 \| \to 0$, and a τ-open set $V \supset P_K(x_0)$ such that $P_K(x_n) \backslash V \neq \emptyset$ for each $n > 0$. Choose $y_n \in P_K(x_n) \backslash V$ for each $n > 0$. Then

$$d(x_0, K) \leqslant \| x_0 - y_n \| \leqslant \| x_0 - x_n \| + \| x_n - y_n \| \\ = \| x_0 - x_n \| + d(x_n, K) \to d(x_0, K).$$

Thus (y_n) is minimizing for x_0. Choose a subnet (resp. subsequence) (y_δ) such that $y_\delta \to^\tau y_0 \in K$. Then (just as in the proof of (1)) $y_0 \in P_K(x_0) \subset V$. But $y_\delta \in X \backslash V$ and $X \backslash V$ is τ-closed so $y_0 \in X \backslash V$, a contradiction.

(3) Let (y_n) be a sequence in $P_K(x)$. Then (y_n) is minimizing for x so it has a subnet (resp. subsequence) τ-converging to a point y_0 in K. By the proof of (1), $y_0 \in P_K(x)$.

(4) Let K be boundedly τ-compact. By Lemma 2.5, (1), (2), and (3) hold. Let $x \in X$ and (y_δ) be a net (resp. sequence) in $P_K(x)$. Since $P_K(x)$ is bounded, there exists a subnet (resp. subsequence) (y_γ) which τ-converges to some $y_0 \in K$. Then (just as in the proof of (1)) $y_0 \in P_K(x)$ so $P_K(x)$ is τ-compact. ▮

In the special case when $\tau = n$ (resp. $\tau = w$), statement (2) of Theorem 2.7 was established by Singer [26] (resp. Vlasov [29]). As noted in the Introduction, Theorem 2.7 was first proved in [6]; also, statements (1) and (3), under the additional assumption that τ be a topology, were established independently by Vlasov [30].

Remark. It is worth noting that we have not yet used the homogeneity property 2.1(iii) in the definition of regular mode of convergence. In fact, we have not even used the full strength of properties 2.1(i) and (ii). More precisely, *all that has been used concerning τ thus far is that if $y_\delta \to^\tau y$ and $x \in X$, then* $\| y - x \| \leqslant \limsup \| y_\delta - x \|$. (This observation is further pursued in Section 6.) In the next lemma, however, explicit use is made of property 2.1(iii).

In the following lemma, and the sequel the closed unit ball in a normed linear space Y will be denoted by $B(Y)$.

2.8. THEOREM. *Let τ be a regular mode of convergence (resp. sequential convergence) on X, and let Y be a τ-closed linear subspace of X. Then $B(Y)$ is τ-compact $\Leftrightarrow$ each τ-closed subset of Y is boundedly τ-compact.*

Proof. Let $B(Y)$ be τ-compact and K a τ-closed subset of Y. Let (x_δ) be a bounded net (resp. sequence) in K. Since $B(Y)$ is τ-compact, so is every multiple $\alpha B(Y)$ (by property 2.1(iii)). Hence there is a subnet (resp. subsequence) which τ-converges to some $x \in X$. Since K is τ-closed, $x \in K$. Thus K is boundedly τ-compact.

For the converse, it suffices to show that $B(Y)$ is τ-closed. Let (y_δ) be a net (resp. sequence) in $B(Y)$ and $y_\delta \to^\tau y$. Since Y is τ-closed, $y \in Y$. By norm domination, $\|y\| \leq \limsup \|y_\delta\| \leq 1$ so $y \in B(Y)$ and the proof is complete. ∎

2.9. COROLLARY. *Let Y be a reflexive subspace of X. Then each weakly closed subset K of Y is boundedly w-compact. In particular, K is proximinal and P_K is norm-weak upper semicontinuous.*

The proof, of course, follows since reflexive spaces are characterized by the weak compactness of their unit balls.

Klee had observed in [19] that reflexive subspaces are proximinal.

Since the unit ball in a dual space is weak* compact, we obtain

2.10. COROLLARY. *Every weak* closed subset K of a dual space is boundedly w^*-compact. In particular, K is proximinal and P_K is norm-w^* upper semicontinuous.*

Phelps [21] apparently was the first to observe that weak* closed subsets of dual spaces are proximinal.

As another immediate consequence of Theorem 2.8, we can actually characterize reflexive Banach spaces.

2.11. COROLLARY. *For a Banach space X, the following statements are equivalent.*

(1) *X is reflexive.*

(2) *Each weakly closed subset of X is boundedly w-compact.*

(3) *Each weakly-sequentially closed subset of X is boundedly ws-compact.*

In particular, each weakly closed (e.g., every closed convex) subset of a reflexive space is proximinal and has a norm-weak upper semicontinuous metric projection.

Proof. The equivalence of (1) and (2) (resp. (3)) follows from Theorem 2.8 by taking τ to be the regular mode of convergence (resp. sequential convergence) generated by the weak topology, and by using the Eberlein–Smulian characterization of reflexive Banach space as those whose unit balls are weakly (resp. weakly sequentially) compact. ∎

The equivalence of the three statements in Corollary 2.11 was also essentially established by Vlasov [28] along with the fact that a boundedly w-compact Chebyshev set has a norm-weak continuous metric projection.

A simple consequence of James's well-known characterization of reflexive Banach spaces [16] is that X is reflexive if and only if each closed hyperplane is proximinal. From this fact and Corollary 2.11, we easily deduce the well-known

2.12. COROLLARY. *For a Banach space X, the following statements are equivalent.*

(1) *X is reflexive.*

(2) *Each weakly closed subset of X is approximatively w-compact.*

(3) *Each weakly closed subset of X is approximatively ws-compact.*

(4) *Each weakly sequentially closed subset of X is proximinal.*

(5) *Each weakly closed subset of X is proximinal.*

(6) *Each closed convex subset of X is proximinal.*

(7) *Each closed subspace of X is proximinal.*

The equivalence of (1), (3), and (4) was first established by Breckner [3]; while the equivalence of (1), (5), (6), and (7) is essentially James' theorem (cf., e.g., [27]).

We next turn to the question of when an approximatively τ-compact set is actually approximatively (norm-) compact.

2.13. DEFINITION. Let τ be a regular mode of convergence (resp. sequential convergence) on X. X is said to have *property* (A_τ) provided that $\| x_\delta - x \| \to 0$ whenever (x_δ) is a net (resp. sequence) with $x_\delta \to^\tau x$ and $\| x_\delta \| \to \| x \|$.

When $\tau = \text{ws}$, the regular mode of sequential convergence generated by the weak topology, property (A_τ) reduces to the well-known property first studied by Kadec [17] and Fan and Glicksberg [11].

2.14. EXAMPLES. (i) Every locally uniformly convex space, hence every uniformly convex space, has property (A_{ws}) and property (A_{w}) (see [4], p. 113).

(ii) Every locally uniformly convex dual space has property $(A_{\text{w*s}})$ and property $(A_{\text{w*}})$.

(iii) The $L_p(\mu)$ spaces, $1 \leqslant p < \infty$, have property $(A_{\text{a.e.}})$ (see, e.g., [14], p. 209.).

The importance of property (A_τ) stems from the following result.

2.15. THEOREM. *In a space which has property (A_τ), every approximatively τ-compact set is approximatively compact.*

Proof. Let τ be a regular mode of convergence (resp. sequential convergence) on X. Let $K \subset X$ be an approximatively τ-compact set, $x \in X$, and let

(y_n) in K be minimizing for x: $\| x - y_n \| \to d(x, K)$. Then (y_n) has a subnet (resp. subsequence) (y_δ) which τ-converges to some $y \in K$. Then $y \in P_K(x)$, $\| x - y \| = d(x, K) = \lim \| x - y_\delta \|$, and $x - y_\delta \to^\tau x - y$. By property (A_τ), $\| y - y_\delta \| = \|(x - y_\delta) - (x - y)\| \to 0$. Thus (y_δ) converges in norm to y so K is approximatively compact. ▌

In the particular case when $\tau =$ ws, Theorem 2.15 was first proved by Breckner [3].

2.16. Corollary. *Let τ be a regular mode of convergence (resp. sequential convergence) on X. If $B(X)$ is τ-compact and X has property (A_τ), then every τ-closed set is approximatively compact.*

Proof. Theorem 2.8, Lemma 2.5, and Theorem 2.15. ▌

2.17. Corollary. *Let X be a locally uniformly convex dual space. Then every weak* closed subset of X is approximatively compact. In particular, every weak* closed convex subset of X is a Chebyshev set with a continuous metric projection.*

Proof. It suffices to observe the following facts and then apply Corollary 2.16: (1) the unit ball in a dual space is weak* compact; (2) a locally uniformly convex dual space has property (A_{w*}); (3) in a strictly convex space, every point has at most one best approximation from a convex set. ▌

The last statement of Corollary 2.17, for weak* closed subspaces, was essentially proved by Lindenstrauss [20] using a selection theorem.

Since the weak and weak* topologies coincide in reflexive spaces, Corollary 2.17 implies the following well-known result of Efimov and Stechkin [10].

2.18. Corollary. *Let X be a reflexive locally uniformly convex Banach space (e.g., a uniformly convex Banach space). Then every weakly closed subset is approximatively compact. In particular, every closed convex subset of X is a Chebyshev set with a continuous metric projection.*

It is known that if τ is generated by the weak topology, then the *converse* of Corollary 2.16 is also valid. Indeed, we have

2.19. Theorem. *Let X be a Banach space. The following statements are equivalent.*

(1) *X is reflexive and has property (A_w).*

(2) *Every weakly closed set is approximatively compact.*

(3) *Every closed convex set is approximatively compact.*

(4) *Every closed hyperplane is approximatively compact.*

The implication (1) $\Rightarrow$ (2) follows from Corollary 2.16, while the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. The implication (4) $\Rightarrow$ (1) follows using James' characterization of reflexive Banach spaces [16]. Theorem 2.19, under the additional assumption that X be strictly convex, was essentially established by Fan and Glicksberg [11]. (Singer [26] observed that strict convexity was not essential to their proof.)

2.20. PROBLEM. Is the converse to Corollary 2.16 valid? That is, if every τ-closed set is approximatively compact must $B(X)$ be τ-compact and X have property (A_τ)?

There is a partial converse of Corollary 2.16 which we now state. For the purposes of this result we need the following definition.

2.21. DEFINITION. A regular mode of sequential convergence τ is called *fully regular* if

(i) "Scalar multiplication is 'τ-continuous,'" i.e., $x_n \to^\tau x$ and (α_n) scalars with $\alpha_n \to \alpha$ implies $\alpha_n x_n \to^\tau \alpha x$.

(ii) $x_n \to^\tau x$ implies $x_{n_k} \to^\tau x$ for each subsequence (x_{n_k}) of (x_n).

(iii) "Limits are unique," i.e., $x_n \to^\tau x$ and $x_n \to^\tau y$ implies $x = y$.

We note that *all* of the examples of regular modes of sequential convergence given in 2.2 are fully regular.

2.22. PROPOSITION. *Let τ be a fully regular mode of sequential convergence on the normed space X. If every τ-closed subset of X is approximatively compact, X has property (A_τ).*

Proof. Let (x_n) in X, $x_n \to^\tau x_0$, and $\|x_n\| \to \|x_0\|$. To show: $\|x_n - x_0\| \to 0$. If $x_0 = 0$, the result is obvious so we assume $x_0 \neq 0$ and let $y_n = x_n/\|x_n\|$ $(n = 0, 1, 2,...,)$. Then $\|y_n\| = 1$ for all n and, by 2.21(i), $y_n \to^\tau y_0$. Let $K = \{y_n \mid n = 0, 1, 2,...,\}$. We first show that K is τ-closed. Let $\{z_n\}$ be in K and $z_n \to^\tau z$. If $\{z_n\}$ is finite, then (z_n) has a constant subsequence: $z_{n_k} = y_{n_0}$ $(k = 1, 2,...,)$. Then by 2.21(ii) $z = z_{n_k} = y_{n_0} \in K$. If $\{z_n\}$ is not finite, it is possible to choose a subsequence (z_{n_k}) so that letting $z_{n_k} = y_{n'_k}$ for some n'_k, $(y_{n'_k})$ will be a subsequence of (y_n). Hence by 2.21(ii) $y_{n'_k} = z_{n_k} \to^\tau z$ and $y_{n'_k} \to^\tau y_0$. By 2.21(iii), $z = y_0 \in K$. Thus is τ-closed. By hypothesis, K is approximatively compact. But

$$\|y_n\| = 1 = d(0, K) \qquad (n = 0, 1, 2,...,)$$

so (y_n) is a minimizing sequence for 0. Hence there is a subsequence (y_{n_k}) and a point $y \in K$ such that $\|y_{n_k} - y\| \to 0$. Since $y_{n_k} \to^\tau y_0$, we have that

$$\|y_0 - y\| \leqslant \limsup \|y_{n_k} - y\| = 0$$

so $y = y_0$. That is, $\| y_{n_k} - y_0 \| \to 0$. This argument shows that every subsequence of (y_n) has a subsequence which converges in norm to y_0. Hence $\| y_n - y_0 \| \to 0$. Thus

$$\| x_n - x_0 \| \leqslant \| x_n \| \left[\| y_n - y_0 \| + \left| \frac{1}{\| x_0 \|} - \frac{1}{\| x_n \|} \right| \| x_0 \| \right] \to 0,$$

and this completes the proof. ▮

Combining Corollary 2.16 with Proposition 2.22 we obtain

2.23. PROPOSITION. *Let τ be a fully regular mode of sequential convergence on the normed space X and suppose $B(X)$ is τ-compact. Then every τ-closed subset of X is approximatively compact $\Leftrightarrow$ X has property (A_τ).*

If X^* is the dual of a *separable* normed linear space X, it is known that the Banach–Alaoglu theorem (i.e., $B(X^*)$ is weak* compact) can be strengthened to: $B(X^*)$ is weak* sequentially compact (w*s-compact). Thus we obtain from Proposition 2.23 the following approximation theoretic characterization of property (A_{w*s}).

2.24. PROPOSITION. *Let X be a separable normed linear space. Then every w*s-closed subset of X^* is approximatively compact $\Leftrightarrow$ X^* has property (A_{w*s}): (x_n^*) in X^*, $\| x_n^* \| = 1$ $(n = 0, 1, 2,...,)$, and $x_n^*(x) \to x_0^*(x)$ for all $x \in X$ implies $\| x_n^* - x_0^* \| \to 0$.*

3. APPLICATIONS IN $C[a, b]$

Let G and H be finite-dimensional subspaces of $C[a, b]$ consisting of analytic functions. We define the "generalized" rational functions by

$$\mathscr{R} = \mathscr{R}(G, H) = \{ r \in C[a, b] \mid rh = g, h \in H \backslash \{0\}, g \in G \},$$

Recall from 2.2.4 that a net (x_δ) in $C[a, b]$ is said to Δ-converge to $x \in C[a, b]$ iff $x_\delta(t) \to x(t)$ for all t in some dense subset of $[a, b]$.

3.1. THEOREM. *$\mathscr{R}$ is boundedly Δ-compact in $C[a, b]$. In particular, $\mathscr{R}$ is proximinal, $P_{\mathscr{R}}$ is norm-Δ u.s.c., and $P_{\mathscr{R}}(x)$ is Δ-compact for each $x \in C[a, b]$.*

Proof. The last statement follows from Theorem 2.7. To prove the first statement, let (r_δ) be a bounded net in $\mathscr{R}$, say $\| r_\delta \| \leqslant M$. Then we can write $r_\delta h_\delta = g_\delta$ for some $h_\delta \in H$, $\| h_\delta \| = 1$, $g_\delta \in G$, and for all $t \in [a, b]$,

$$| g_\delta(t) | = | r_\delta(t)\, h_\delta(t) | \leqslant M \, | h_\delta(t) | \leqslant M. \qquad (*)$$

That is, $\| g_\delta \| \leqslant M$ for all δ. Since G and H are finite dimensional, and (h_δ) and (g_δ) are bounded, by passing to a subnet, we may assume $\| g_\delta - g_0 \| \to 0$ for some $g_0 \in G$ and $\| h_\delta - h_0 \| \to 0$ for some $h_0 \in H$, $\| h_0 \| = 1$. Now h_0 can have only finitely many zeros and passing to the limit in Eq. (*) we get

$$| g_0(t)| \leqslant M \, | h_0(t)| \qquad (t \in [a, b]). \tag{**}$$

Hence each zero of h_0 is a zero of g_0. Thus we see that the function $r_0 = g_0/h_0$ is well defined and continuous on $[a, b]\backslash Z(h_0)$, where $Z(h_0)$ is the zero set of h_0. Further, no matter how r_0 is defined on $Z(h_0)$, $r_0 h_0 = g_0$. If we cancel the common zero factors of h_0 and g_0 on $Z(h_0)$, r_0 is seen to be well defined and continuous everywhere. Thus $r_0 \in C[a, b]$ and $r_0 h_0 = g_0$, i.e., $r_0 \in \mathscr{R}$. Furthermore, if $t \in [a, b]\backslash Z(h_0)$,

$$r_0(t) = \frac{g_0(g)}{h_0(t)} = \lim \frac{g_\delta(t)}{h_\delta(t)} = \lim r_\delta(t).$$

That is, $r_\delta \to^{\Delta} r_0$. ∎

The "existence" part of Theorem 3.1 is essentially due to Walsh [31]. Note that the proof of Theorem 3.1 actually shows that $\mathscr{R}$ is boundedly φ-compact.

The "ordinary" rational functions are defined by

$$\mathscr{R}_m{}^n = \left\{ \frac{g}{h} \,\middle|\, g \in \mathscr{P}_n, h \in \mathscr{P}_m, h > 0 \text{ on } [a, b] \right\},$$

where $\mathscr{P}_k$ denotes the set of all polynomials of degree at most k.

3.2. COROLLARY. *$\mathscr{R}_m{}^n$ is boundedly Δ-compact in $C[a,b]$. In particular, $\mathscr{R}_m{}^n$ is proximinal, $P_{\mathscr{R}_m^n}$ is norm-Δ u.s.c., and $P_{\mathscr{R}_m^n}(x)$ is Δ-compact for each $x \in C[a, b]$.*

Proof. We need only observe that

$$\mathscr{R}_m{}^n = \{r \in C[a, b] \mid rh = g \text{ for some } h \in \mathscr{P}_m \backslash \{0\}, g \in \mathscr{P}_n\}$$

and apply Theorem 3.1. ∎

The *exponential sums of order N* form the subset of $C[a, b]$ defined by

$$E_N = \left\{ \sum_{i=1}^{l} p_i(t) \, e^{\lambda_i t} \,\middle|\, p_i \text{ is a polynomial of degree } \partial p_i, \lambda_i \in \mathbb{R}, \text{ and } \sum_1^l (\partial p_i + 1) \leqslant N \right\}.$$

The spline functions of order n with k free knots form the subset of $C[a, b]$ defined by

> $S_{n,k} = \{x \in C[a, b] \mid$ there exist $a = t_0 < t_1 < \cdots < t_{r+1} = b$ and integers $m_1, ..., m_r$ in $\{1, 2, ..., n+1\}$ with $\sum_1^r m_i = k$ such that $x \in \mathscr{P}_n$ in each interval (t_i, t_{i+1}), while x has continuous derivatives of order $n - m_i$ in a neighborhood of t_i $(i = 1, 2, ..., r)\}$.

3.3. THEOREM. *Let $K = E_n$ or $S_{n,k}$. Then K is boundedly Δs-compact in $C[a, b]$. In particular, K is proximinal, P_K is norm $-\Delta s$ u.s.c., and $P_K(x)$ is Δs-compact for each $x \in C[a, b]$.*

Proof. First let $K = E_n$. Let (y_n) be a bounded sequence in K. By a result of Werner [32], there exists a subsequence (y_{n_k}) which converges pointwise, except possibly at the end points of $[a, b]$, to some $y_0 \in K$. Thus $y_n \to^{\Delta} y_0$ and K is boundedly Δs-compact.

Next let $K = S_{n,k}$. A close inspection of Schumaker's proof [25] that K is proximinal reveals that he actually showed that every bounded sequence in K has a subsequence which converges pointwise, except possibly at a finite number of points, to an element of K. Thus K is boundedly Δs-compact. ∎

The following example shows that the notions of approximative (norm-) compactness and approximative weak sequential-compactness do not suffice for proving existence theorems.

3.4. *Example of a Subset of* $C[0, 1]$ *Which is Boundedly ΔS-Compact but not Approximatively Weak Sequentially-Compact.*

Such an example is, of course, proximinal (by Theorem 2.7) and is not approximatively (norm-) compact since the norm topology is stronger than the weak topology.

Let $K = \{0\} \cup \{F(a) \mid 0 \leqslant a < \infty\}$, where

$$F(a)(t) = \frac{1}{at + 1} \qquad (0 \leqslant t \leqslant 1).$$

Since $\| F(a) - F(b) \| \leqslant | a - b |$, F is Lipschitz continuous. Let $\{F(a_n)\}_1^\infty$ be a sequence in K. If $\{a_n \mid n = 1, 2, ...\}$ is bounded, choose a subsequence (a_{n_j}) which converges to some $a \in [0, \infty)$. Then $\| F(a_{n_j}) - F(a) \| \leqslant | a_{n_j} - a | \to 0$ and this implies that $F(a_{n_j}) \to^{\Delta} F(a)$. If $\{a_n \mid n = 1, 2, ...\}$ is unbounded, choose a subsequence $a_{n_j} \to \infty$. Then

$$F(a_{n_j})(t) \to g(t) = 1 \quad \text{if} \quad t = 0,$$
$$= 0 \quad \text{if} \quad 0 < t \leqslant 1,$$

and this implies that $F(a_{n_j}) \to^{\Delta} 0$. In either case, $(F(a_{n_j}))$ Δ-converges to an element of K. Thus K is boundedly Δs-compact.

To see that F is not approximatively weak-compact, observe that if x is the constant function $x(t) = \frac{1}{2}$, then $(F(n))_{n=1}^{\infty}$ is a minimizing sequence for x which converges pointwise to the function g defined above. If $(F(n))_1^{\infty}$ had a subnet converging weakly to some $f \in C[0, 1]$, then it must also converge pointwise to f, which implies $f = g \notin C[0, 1]$, an absurdity.

4. Applications in $L_p[a, b]$, $1 \leq p \leq \infty$

Fix any p with $1 \leqslant p \leqslant \infty$ and let G and H be finite-dimensional subspaces of $L_p[a, b]$ consisting of analytic functions. Consider the "generalized rational" functions in $L_p[a, b]$:

$$\mathscr{R} = \mathscr{R}(G, H) = \{r \in C[a, b] \mid rh = g, h \in H\backslash\{0\}, g \in G\}.$$

Recall that for a sequence (x_n) in $L_p[a, b]$ we write $x_n \to^{\text{a.e.}} x$ if x_n converges to x almost everywhere.

The following result was obtained in collaboration with R.E. Huff.

4.1. Theorem. (1) *If* $1 \leqslant p < \infty$, $\mathscr{R}$ *is approximatively compact. In particular,* $\mathscr{R}$ *is proximinal,* $P_{\mathscr{R}}$ *is norm-norm upper semicontinuous, and* $P_{\mathscr{R}}(x)$ *is compact for each* $x \in L_p[a, b]$.

(2) $\mathscr{R}$ *is boundedly* a.e.*-compact in* $L_\infty[a, b]$. *In particular,* $\mathscr{R}$ *is proximinal,* $P_{\mathscr{R}}$ *is norm-*a.e. *upper semicontinuous, and* $P_{\mathscr{R}}(x)$ *is* a.e.*-compact for each* $x \in L_\infty[a, b]$.

Proof. (1) We first show that $\mathscr{R}$ is boundedly a.e.-compact in $L_p[a, b]$ $1 \leqslant p < \infty$. Let (r_n) be a bounded sequence in $\mathscr{R}$: $\| r_n \|_p \leqslant M$. We have that

$$r_n h_n = g_n \qquad \text{for some } h_n \in H\backslash\{0\}, g_n \in G. \tag{*}$$

By scaling both sides of this equation, we may assume $\| h_n \|_q = 1$, where $1/p + 1/q = 1$. Using (*) and Hölder's inequality, we obtain

$$\| g_n \|_1 = \int_{[a,b]} | r_n h_n | \leqslant \| r_n \|_p \| h_n \|_q \leqslant M.$$

Since G and H are finite dimensional there is a subsequence (n_k) of the natural numbers and points $g_0 \in G$, $h_0 \in H$ such that

$$\| g_{n_k} - g_0 \|_1 \to 0 \qquad \text{and} \qquad \| h_{n_k} - h_0 \|_q \to 0.$$

Since all norms are equivalent on a finite-dimensional space, we have $\| g_{n_k} - g_0 \|_\infty \to 0$ and $\| h_{n_k} - h_0 \|_\infty \to 0$. Clearly $\| h_0 \|_q = 1$ so h_0 has at most finitely many zeros since it is analytic. Thus except on the (finite) zero set $Z(h_0)$, we have

$$r_{n_k}(t) = \frac{g_{n_k}(t)}{h_{n_k}(t)} \to \frac{g_0(t)}{h_0(t)}.$$

Now g_0/h_0 is well defined and continuous except on $Z(h_0)$. Thus by defining g_0/h_0 to be constantly 1 on $Z(h_0)$, it follows that g_0/h_0 is measurable (cf. [14; Theorems 11.8(v) and 11.11]). Further, by Fatou's lemma,

$$\begin{aligned}\int_{[a,b]} | x - g_0/h_0 |^p &\leqslant \liminf \int_{[a,b]} | x - r_{n_k} |^p \\ &\leqslant \liminf [\| x \| + \| r_{n_k} \|]^p \leqslant [\| x \| + M]^p\end{aligned}$$

implies $\| x - g_0/h_0 \|_p \leqslant \| x \| + M$ and, in particular $g_0/h_0 \in L_p[a, b]$.

We now show that there exists $r_0 \in \mathscr{R}$ such that $r_{n_k} \to^{\text{a.e.}} r_0$. Let t_0 be a zero of h_0 in $[a, b]$ with multiplicity μ:

$$h_0(t) = (t - t_0)^\mu h(t), \qquad h(t_0) \neq 0.$$

If g_0 does not have a zero at t_0 of multiplicity $\geqslant \mu$, then in some neighborhood of t_0 we have

$$\frac{g_0(t)}{h_0(t)} = \frac{g(t)}{(t - t_0)^\nu h(t)},$$

where g, h are analytic, $h(t_0) \neq 0 \neq g(t_0)$, and $\nu \geqslant 1$. Thus for all t in some (perhaps smaller) neighborhood E of t_0, we have

$$\left| \frac{g(t)}{h(t)} \right| \geqslant \delta > 0.$$

Since $g_0/h_0 \in L_p[a, b]$, we get

$$\infty > \int_E \left| \frac{g_0(t)}{h_0(t)} \right|^p dt \geqslant \delta^p \int_E \frac{1}{| t - t_0 |^{\nu p}} dt = \infty$$

because $\nu p \geqslant 1$. This contradiction shows that every zero of h_0 is also a zero of g_0 with at least as large multiplicity. Thus the function g_0/h_0 may be redefined at the zeros of h_0 (i.e., on $Z(h_0)$) so that the resulting function r_0 is continuous on $[a, b]$. Clearly, $h_0 r_0 = g_0$ so $r_0 \in \mathscr{R}$ and $r_{n_k} \to^{\text{a.e.}} r_0$.

This proves that $\mathscr{R}$ is boundedly a.e.-compact in $L_p[a, b]$. By Lemma 2.5, $\mathscr{R}$ is approximatively a.e.-compact. As noted in 2.14(iii), $L_p[a, b]$ has property

($A_{\text{a.e.}}$). By Theorem 2.15, $\mathscr{R}$ is approximatively compact. The last statement of part (1) follows from Theorem 2.7.

(2) The proof of the case $p = \infty$ is exactly the same as in Theorem 3.1. ∎

The fact that $\mathscr{R}$ is approximatively compact in $L_p[a, b]$ for $1 < p < \infty$, also follows from a result of Blatter [2] who used a different proof. See also Wolfe [33] for a related result.

Just as in Section 3, if we specialize by taking $G = \mathscr{P}_n$, $H = \mathscr{P}_m$ (the polynomials of degree at most n and m, respectively), we get the "ordinary rational functions"

$$\mathscr{R}_m{}^n = \left\{ \frac{g}{h} \,\middle|\, g \in \mathscr{P}_n,\ h \in \mathscr{P}_m,\ h > 0 \text{ on } [a, b] \right\}.$$

Thus Theorem 4.1 implies

4.2. Corollary. (1) *If* $1 \leqslant p < \infty$, *then* $\mathscr{R}_m{}^n$ *is approximatively compact in* $L_p[a, b]$. *In particular,* $\mathscr{R}_m{}^n$ *is proximinal,* $P_{\mathscr{R}_m^n}$ *is norm-norm upper semicontinuous, and* $P_{\mathscr{R}_m^n}(x)$ *is compact for each* $x \in L_p[a, b]$.

(2) $\mathscr{R}_m{}^n$ *is boundedly* a.e.*-compact in* $L_\infty[a, b]$. *In particular,* $\mathscr{R}_m{}^n$ *is proximinal,* $P_{\mathscr{R}_m^n}$ *is norm-*a.e. *upper semicontinuous, and* $P_{\mathscr{R}_m^n}(x)$ *is* a.e.*-compact for each* $x \in L[a, b]$.

Hobby and Rice [15] (see also Rice [22, pp. 46–53]) have shown that a certain class of "γ-polynomials" is boundedly compact in $L_p[0, 1]$, $1 \leqslant p \leqslant \infty$. The proof given in [15] holds only if the nonlinear parameters come from a compact set. An alternate approach, using divided differences, which is valid in the general case, was given by de Boor [5]. (In particular, the ordinary polynomials and exponential sums are included as special cases.) Thus for $\tau = n$, the norm topology, the conclusion of Theorem 2.7 holds for the "γ-polynomials" in $L_p[0, 1]$, $1 \leqslant p \leqslant \infty$.

Efimov and Stechkin [10] proved that every approximatively compact Chebyshev set in a uniformly convex space must be convex. It follows that in $L_p[a, b]$, $1 < p < \infty$, none of the sets $\mathscr{R}$, $\mathscr{R}_n{}^m$, or the exponential sums is a Chebyshev set (except in trivial cases).

5. Applications in the Space of Operators

Let X and Y be normed linear spaces, $\mathscr{L}(X, Y)$ the space of all bounded linear operators L from X into Y endowed with the operator norm: $\| L \| = \sup\{\| Lx \| \mid x \in X, \| x \| \leqslant 1\}$, and let $B(\mathscr{L}(X, Y))$ denote the closed unit ball in $\mathscr{L}(X, Y)$. The weak and weak* operator topologies on $\mathscr{L}(X, Y)$ (as described in 2.2.10 and 2.2.11) are regular modes of convergence.

The following useful lemma can be proved in a manner similar to Alaoglu's theorem [8, p. 424].

5.1. LEMMA. *$B(\mathscr{L}(X, Y))$ is wo-closed. If Y is a dual space, $B(\mathscr{L}(X, Y))$ is w*o-compact. If Y is reflexive, $B(\mathscr{L}(X, Y))$ is wo-compact.*

5.2. THEOREM. *Let Y be a dual space (resp. reflexive) and let $\mathscr{K}$ be a w*o-closed (resp. wo-closed) subset of $\mathscr{L}(X, Y)$. Then $\mathscr{K}$ is boundledly w*o-compact (resp. boundedly wo-compact). In particular, $\mathscr{K}$ is proximinal, $P_{\mathscr{K}}$ is norm-w*o (resp. norm-wo) upper semicontinuous, and $P_{\mathscr{K}}(L)$ is w*o-compact (resp. wo-compact) for each $L \in \mathscr{L}(X, Y)$.*

Proof. From Lemma 5.1, every norm-closed ball B in $\mathscr{L}(X, Y)$ is w*o-compact (resp. wo-compact). Hence $B \cap \mathscr{K}$ is w*o-compact (resp. wo-compact). Thus $\mathscr{K}$ is boundedly w*o-compact (resp. boundedly wo-compact). The last statement of the theorem follows from Theorem 2.7. ∎

5.3. COROLLARY. *Let X be a Hilbert space and let $\mathscr{K}$ be a wo-closed subset of $\mathscr{L}(X, X)$ (e.g., $\mathscr{K}$ is the set of all positive operators, or $\mathscr{K}$ is the set of all Hermitian operators). Then $\mathscr{K}$ is boundedly wo-compact. In particular, $\mathscr{K}$ is proximinal, $P_{\mathscr{K}}$ is norm-wo upper semicontinuous, and $P_{\mathscr{K}}(L)$ is wo-compact for each $L \in \mathscr{L}(X, X)$.*

The existence part of this corollary was first observed, among other things, by Halmos [13].

Rogers [24] has shown that many subsets of the normal operators on a Hilbert space are *not* proximinal (e.g., the normal operators, the compact normal operators, the unitary operators, and the projection operators). He conjectures (in our terminology) that a subset of the normal operators is proximinal if and only if it is boundedly wo-compact.

6. A GENERALIZATION

6.1. DEFINITIONS. Let K and Y be subsets of a normed linear space X. Suppose that certain bounded nets (resp. sequences) (k_δ) in K are said to τ-converge, written $k_\delta \to^\tau k$. Suppose also that this convergence has the following property: if $k_\delta \to^\tau k$ and $y \in Y$, then

$$\| k - y \| \leqslant \limsup \| k_\delta - y \| . \tag{6.1.1}$$

In this situation, we say that τ is a *mode of convergence (resp. sequential convergence) on K relative to Y.*

Let τ be a mode of convergence (resp. sequential convergence) on K

relative to Y. We say that K is *approximatively τ-compact relative to Y* if, for each $y \in Y$, each minimizing sequence in K for y has a subnet (resp. subsequence) which τ-converges to a point in K.

Note that if τ is a regular mode of convergence (resp. sequential convergence) (as defined in 2.1) and K is approximatively τ-compact (as defined in 2.3), then for any subset Y it is obvious that: (1) τ is a mode of convergence (resp. sequential convergence) on K relative to Y; and (2) K is approximatively τ-compact relative to Y. Thus these relativized notions are more general than the original ones given in Section 2.

By inspecting the proof of Theorem 2.7(1), one sees that we have actually verified the following result.

6.1. THEOREM. *Let K and Y be subsets of X and let τ be a mode of convergence (resp. sequential convergence) on K relative to Y. If K is approximatively τ-compact relative to Y, then each $y \in Y$ has a best approximation in K.*

As an application of this theorem, let T be a topological space, $B(T)$ the space of all bounded real-valued functions x on T with the norm $\|x\| = \sup_{t\in T} |x(t)|$, and let $C(T)$ denote the subset of $B(T)$ consisting of all the continuous functions.

We will say that a bounded sequence (x_n) in $B(T)$ d-*converges* to some $x \in B(T)$, and write $x_n \to^{\mathrm{d}} x$, provided that (x_n) converges to x pointwise on some dense subset S of T, and for each $t \in T\backslash S$, the inequality

$$\liminf_{\substack{s\to t\\ s\in S}} x(s) \leqslant x(t) \leqslant \limsup_{\substack{s\to t\\ s\in S}} x(s) \tag{D}$$

holds.

6.2. LEMMA. *d is a mode of sequential convergence on $B(T)$ relative to $C(T)$.*

Proof. Let (x_n) be a bounded sequence in $B(T)$, $x \in B(T)$, and $x_n \to^{\mathrm{d}} x$. Given $y \in C(T)$, we must show that $\|x - y\| \leqslant \limsup \|x_n - y\|$. Since $x_n \to^{d} x$, there exists a dense subset S of T such that $x_n(s) \to x(s)$ for all $s \in S$, and for each $t \in T\backslash S$,

$$\liminf_{\substack{s\to t\\ s\in S}} x(s) \leqslant x(t) \leqslant \limsup_{\substack{s\to t\\ s\in S}} x(s). \tag{*}$$

If $s \in S$,

$$|x(s) - y(s)| = \lim_n |x_n(s) - y(s)| \leqslant \limsup_n \|x_n - y\|.$$

If $t \in T\backslash S$, then using the continuity of y and inequality (*), we obtain

$$\liminf_{\substack{s\to t\\ s\in S}}[x(s) - y(s)] \leqslant x(t) - y(t) \leqslant \limsup_{\substack{s\to t\\ s\in S}}[x(s) - y(s)].$$

Thus

$$\begin{aligned}|x(t)-y(t)| &\leqslant \max\{|\liminf_{\substack{s\to t\\ s\in S}}[x(s)-y(s)]|, \ |\limsup_{\substack{s\to t\\ s\in S}}[x(s)-y(s)]|\}\\ &\leqslant \limsup_{\substack{s\to t\\ s\in S}} |x(s)-y(s)|\\ &= \limsup_{\substack{s\to t\\ s\in S}}\{\lim_n |x_n(s)-y(s)|\}\\ &\leqslant \limsup_n \|x_n-y\|.\end{aligned}$$

It follows that $\|x-y\| \leqslant \limsup \|x_n-y\|$ and the proof is complete. ∎

6.3. Definition (Dunham [9]). A subset K of $B(T)$ is called *dense compact* if each bounded sequence in K has a subsequence converging pointwise on a dense subset S of T to some function $k \in K$ such that, for every $t \in T\backslash S$,

$$\liminf_{\substack{s\to t\\ s\in S}} k(s) \leqslant k(t) \leqslant \limsup_{\substack{s\to t\\ s\in S}} k(s). \tag{6.3.1}$$

Note that inequality (6.3.1) is always satisfied for each *continuous* function k. In particular, a subset of $C(T)$ is dense compact if and only if it boundedly Δ-compact (see 2.2.4).

6.4. Proposition. *If K is a dense compact subset of $B(T)$, then K is approximatively d-compact relative to $C(T)$. In particular, each $x \in C(T)$ has a best approximation in K.*

Proof. By Lemma 6.2, d is a regular mode of sequential convergence on K relative to $C(T)$. Let $x \in C(T)$ and let (k_n) in K be a minimizing sequence for x. Then (k_n) is bounded so by dense compactness it has a subsequence which d-converges to a point in K. Thus K is approximatively d-compact relative to $C(T)$. The last statement follows from Theorem 6.1. ∎

The last statement of Proposition 6.4 was first proved by Dunham [9], who also noted that many of classical approximating families in $C[a, b]$ (e.g., the finite-dimensional subspaces, the rational functions $\mathscr{R}_n{}^m[a, b]$, and the exponential sums of order n) are dense compact. This also follows since a subset of $C(T)$ is dense compact if and only if it is boundedly Δ-compact, and then applying Corollary 3.2 and Theorem 3.3.

We have seen that every approximatively τ-compact set is proximinal. More generally, every set K in X which is approximatively τ-compact relative to X is proximinal. It is natural to ask whether the converse is true. More precisely, if K is a proximinal subset of a normed linear space X, must K be approximatively τ-compact for some mode of convergence (or sequential convergence) τ on K relative to X? The following surprising example, due to Dan Amir (private communication), shows that the answer is negative.

Example of a Chebyshev Hyperplane which is not Approximatively τ-Compact for any τ

Consider the space X of convergent sequences (i.e., $x(\infty) = \lim x(n)$ exists for every $x \in X$) with the norm

$$\|x\| = \max\{2\,|x(0)|, |x(\infty)|\} + \left[\sum_1^\infty \left(\frac{x(k)}{k}\right)^2\right]^{1/2}.$$

Let $M = \{x \in X \mid x(0) = 0\}$. Then M is a Chebyshev hyperplane with linear metric projection given by $P_M(x) = \sum_{k=1}^\infty x(k)\, e_k$, where e_k is the kth unit vector: $e_k(j) = \delta_{kj}$. Let $x = \sum_1^\infty e_k$ and $x_n = \sum_1^n e_k$. Then $x_n \in M$, $P_M(x) = \sum_1^\infty e_k$, and $\|x - x_n\| \to 2 = d(x, M)$, while

$$\|P_M(x)\| = 1 + \left[\sum_1^\infty \frac{1}{k^2}\right]^{1/2} > 1 + \|x_n\|$$

for every n. This shows that (x_n) is a minimizing sequence for x in M for which

$$\overline{\lim_{n\to\infty}}\, \|x_n\| \leqslant \|P_M(x)\| - 1 < \|P_M(x)\|.$$

In view of (6.1.1), this precludes any subnet of (x_n) from τ-converging to $P_M(x)$ whatever the mode of convergence τ relative to X might be.

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